space is locally flat. Our expression for energy is therefore in conformity with criterion (d) as well.

IV. CONCLUSION

Starting from the viewpoint that the expression for energy should be represented by the generator of translations of a preferred set of space-like coordinate surfaces, we were led to the introduction of the minimal vector fields as preferred descriptors. The expression for energy density so obtained has the virtue of being generally applicable, of yielding a positive-definite value for the energy, of corresponding to the preferred energy expressions of Lorentz-covariant theories, and of being in conformity with Mach's principle to the extent that we may conclude that empty spaces (i.e., those for which the total energy vanishes) are locally flat.

The energy density constructed in this paper still has one essential defect in our opinion. In a local neighborhood there is a high degree of arbitrariness in the construction of families of space-like minimal surfaces, and a consequent high degree of arbitrariness in the definition of our energy density. We can attempt to reduce this arbitrariness by employing appropriate boundary conditions. In fact, we know from the work of reference 10 that for asymptotically flat space-times it is essential that the descriptor fields be asymptotically semi-Killing if our "generalized energy" is to be well defined, and is to coincide in value with the usual expression for the mass of asymptotically Schwarzschild solutions. However, it is evident that such attempts to reduce the arbitrariness of the energy expression lead to definitions which are necessarily nonlocal. Perhaps this is the best that one can hope for, but certainly a reasonably unique local definition of energy density would have been preferable.

For spatially closed space-times it is evident from Eqs. (2) and (3) that the total generalized energy vanishes. We thus see that global families of closed minimal surfaces with $\beta_2=0$ can only be found for the (trivial) locally flat space-times. This may be regarded by some as a serious drawback of our construction. An expression suitable for spatially closed space-times will be developed in a subsequent paper.

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Regge Trajectory in Field Theory*

LU SUN LIU† AND K. TANAKA
Argonne National Laboratory, Argonne, Illinois

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The form of the Regge trajectory, the graph of the complex angular momentum $\alpha(t)$ as a function of the square of the momentum transfer t, is studied in field theory within the framework of the ladder approximation in the crossed channel. To order g^2, the trajectory is unbounded in the region $t\approx 4m^2$, but inclusion of the higher-order terms in g^2 removes the divergence and leads to a smooth trajectory that resembles that expected from potential scattering. The various relations between g^2, $\alpha(t)$, and the exchanged mass are discussed.

1. INTRODUCTION

MANY attempts have been made to correlate high-energy experimental data in terms of Regge poles which are generalized bound states and resonances in the complex angular-momentum plane.[1] There is no rigorous proof of the existence of Regge poles in relativistic field thory. Therefore, information on the possible trajectory [in particular, the energy dependence of the complex angular momentum $\alpha(t)$ in field theory] is inferred from potential scattering, where the existence of Regge poles rests on a secure foundation.[2]

Several authors[3,4] have demonstrated that it is possible to obtain some information on the Regge trajectory of $\alpha(t)$ in field theory, within the framework of the ladder approximation in the crossed channel.

The purpose of this paper is to re-examine the possible Regge trajectory in field theory, within the ladder approximation. A comparison of the present approach with perturbation theory shows that the higher-order terms in the coupling constant make very important contributions to the trajectory. A simple derivation of the trajectory equation is given in Sec. 2. The trajectory is discussed in Sec. 3, the case $\alpha(0)$ is treated in Sec. 4, and finally a summary is given in Sec. 5.

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† Also at Northwestern University, Evanston, Illinois.

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[2] T. Regge, Nuovo Cimento 14, 951 (1959); 18, 947 (1960).

[3] L. Bertocchi, S. Fubini, and M. Tonin, Nuovo Cimento 25, 626 (1962), hereafter called BFT. This contains references to earlier work.

[4] B. W. Lee and R. F. Sawyer, Phys. Rev. 127, 2266 (1962).

2. DERIVATION OF THE EQUATION FOR THE REGGE TRAJECTORY

Consider first the elastic scattering amplitude of scalar pions of mass m and momenta p and q into those of p' and q' in the s channel. The extension to other cases is straightforward. In this model it is assumed that the dominant contribution to the imaginary part of the scattering amplitude comes from the sum over all the ladder graphs of Fig. 1 in the t channel, where the rungs represent a quantity of total mass $(s_0)^{1/2}$ with total angular momentum zero.

The square of the total energy is $s=(p+q)^2$ and the square of the momentum transfer is $t=(q-q')^2$ in the s channel. The imaginary part of the scattering amplitude in Fig. 1 can be written as

$$A(s,t)=\sum_{n=0}^{\infty} A_n(s,t).$$

In the forward direction $t=0$, $A(s,0)$ is related to the total cross section through the optical theorem.

In the asymptotic limit $s\to\infty$, the amplitude $A(s,t)$ satisfies the homogeneous integral equation[3]

$$A(s,t)=\int ds_0 \int_0^s \frac{ds'}{s}\int \frac{dk^2dk'^2\, A(s',t,k^2,k'^2)}{(k^2-m^2)(k'^2-m^2)} \times K(s,s',t,s_0,k^2,k'^2), \quad (1)$$

where

$$K(s,s',t,s_0,k^2,k'^2) = \frac{1}{(2\pi)^4}\int d^2n\, \delta\left[(r+n)^2+k^2-m^2\frac{s'}{s}+\frac{s_0s'}{s-s'}\right] \times\delta\left[(r-n)^2+k'^2-m^2\frac{s'}{s}+\frac{s_0s'}{s-s'}\right]A_0(s_0,k^2,k'^2), \quad (2)$$

$$r^2=-t(s-s')/4s,$$

$$A_0(s_0)=\pi g^2 f(s_0).$$

The units have been so chosen that $\hbar=c=1$.

The inhomogeneous term $A_0(s)$, which contributes to the low-energy scattering, is assumed to go to zero faster than the remainder and has been dropped. In this model, unitarity in the s channel is violated, but it is satisfied in the t channel for $4m^2\leqslant t\leqslant 16m^2$. The δ functions in Eq. (2) express the boundaries of the phase space which were found by BFT.

It will now be assumed that $A_0(s_0,k^2,k'^2)$ is not changed from its physical amplitude $A_0(s_0)$, although the two incoming pions are off the mass shell. If the scattering amplitude in the s channel is dominated at high energy by a single Regge pole of angular momentum α, then as $s\to\infty$

$$A(s,t)\to s^{\alpha(t)}\phi(t). \quad (3)$$

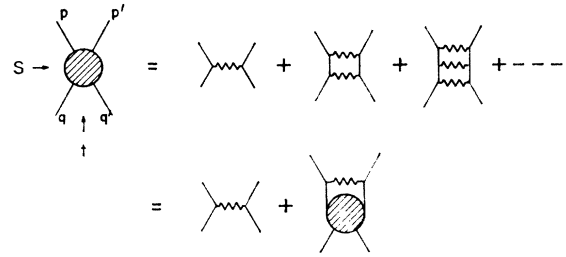

FIG. 1. Ladder graphs for the imaginary part of the scattering amplitude.

It is here inferred from potential scattering that α depends on t. This t is the squared-momentum transfer (for $t\leqslant 0$) in the s channel and also is the square of the total energy (for $t\geqslant 0$) in the t channel. We extract out this important factor $s^{\alpha(t)}$. The integral equation (1) admits solutions of this form.[3]

From the energy denominators of Eq. (1), it is clear that the important contribution comes from the low values of k^2 and k'^2. Further, $A(s',t,k^2,k'^2)$ is dominated by the factor $s'^{\alpha(t)}$ and is expected to be insensitive to the values of k^2 and k'^2 in the high-energy limit, so that k^2 and k'^2 may be put on the mass shell.[5] Consequently, let us examine the solution of Eq. (1) for the case in which

$$A(s',t,k^2,k'^2)\to s'^{\alpha(t)}\phi(t,m^2,m^2)=s^{\alpha(t)}\phi(t). \quad (4)$$

The range of integration of s' is from 0 to s. Therefore, it is assumed that Eq. (4) can be used even outside the asymptotic region. The approximation of putting the pion momenta on the mass shell differs from standard perturbation theory in that an expansion in terms of the coupling constant is not implied. Its relationship with perturbation theory is discussed later. The high-energy limit of the contribution from the two lowest graphs in Fig. 1 agrees with this approximation.

After Eqs. (3) and (4) are substituted into Eq. (1) and the integration over k^2 and k'^2 has been carried out, the integration variable s' is changed to $x=s'/s$. The result is

$$1=\frac{g^2}{32\pi^2}\int ds_0\, f(s_0)\int_{-1}^{1}dz\int_0^1 dx\, x^{\alpha}\left[-\frac{t}{4}(1-z^2) \times(1-x)+\frac{s_0x}{1-x}+m^2(1-x)\right]^{-1}. \quad (5)$$

This interesting equation, which relates $\alpha(t)$ with g^2, has been derived from Eq. (1) in BFT by using a spectral representation of $\phi(t,k^2,k'^2)$ and taking the lowest-order approximation of an iteration method. It can be shown by differentiating Eq. (5) that $d\alpha(t)/dt>0$ for $t\leqslant 4$, and that $\alpha(t)$ becomes complex when $t\geqslant 4$.

[5] The quantities k^2 (and k'^2) are bounded in the integrand by $m^2\leqslant -(k^2-m^2)\leqslant\infty$, etc. We have put $k^2-m^2=0$, $k'^2-m^2=0$ in $A(s',t,k^2,k'^2)$.

One can see from Eq. (5) that (a) when g^2 is fixed, an increase in s_0 leads to a smaller $\alpha(t)$ for $\alpha(t)>0$, $t\leqslant 4m^2$; (b) when $\alpha(t)$ is fixed, an increase in s_0 leads to a larger value of g^2; (c) when s_0 is fixed, an increase in g^2 leads to a larger value of $\alpha(t)$ for $\alpha(t)>0$, $t\leqslant 4m^2$; and (d) a larger value of g^2 leads to a lower resonant energy t_R, i.e., to a larger binding energy. All of these features are expected from a physical point of view.

3. DISCUSSION OF THE REGGE TRAJECTORY

The trajectory, Eq. (5), can be rewritten as

$$1=\frac{g^2}{32\pi^2}\int ds_0\, f(s_0)\int_{-1}^{1}\frac{dz}{-\frac{1}{4}t(1-z^2)+m^2}\times\int_0^1 dx\,\frac{x^\alpha-x^{\alpha+1}}{x^2-2x\cos\lambda+1},\quad(6)$$

where

$$\cos\lambda=1-\frac{s_0}{2[-\frac{1}{4}t(1-z^2)+m^2]}.\quad(7)$$

Integration over x yields

$$1=\frac{g^2}{32\pi^2}\int ds_0\, f(s_0)\int_{-1}^{1}\frac{dz}{-\frac{1}{4}t(1-z^2)+m^2}\times\sum_{n=1}^{\infty}\frac{\sin n\lambda}{\sin\lambda}\left[\frac{1}{n+\alpha}-\frac{1}{n+\alpha+1}\right].\quad(8)$$

This clearly shows that there is a singularity, whenever α is a negative integer. The series converges for $\alpha(t)>-1$. We shall separate out the singularity at $\alpha=-1$ and shall not consider the others.

Consider now the simple case in which only a discrete squared mass a is exchanged, i.e., the case $f(s_0)=\delta(s_0-a)$. Integration over s_0, separation of the singularity at $\alpha=-1$, and rearrangement of the remainder (which is regular at $\alpha=-1$) yields

$$\alpha+1=\frac{g^2}{32\pi^2}\int_{-1}^{1}\frac{dz}{-\frac{1}{4}t(1-z^2)+m^2}+\frac{(\alpha+1)g^2}{32\pi^2}\sum_{n=1}^{\infty}\frac{R_n(t,a)}{n+\alpha+1},\quad(9)$$

where

$$R_n(t,a)=\int_{-1}^{1}dz\frac{[\sin(n+1)\lambda-\sin n\lambda]/\sin\lambda}{-\frac{1}{4}t(1-z^2)+m^2}.$$

Equation (9) can be rewritten as

$$\alpha+1=\frac{g^2}{32\pi^2}\left\{\int_{-1}^{1}\frac{dz}{-\frac{1}{4}t(1-z^2)+m^2}\Bigg/\left[1-\frac{g^2}{32\pi^2}\sum_{n=1}^{\infty}\frac{R_n(t,a)}{n+\alpha+1}\right]\right\}.\quad(10)$$

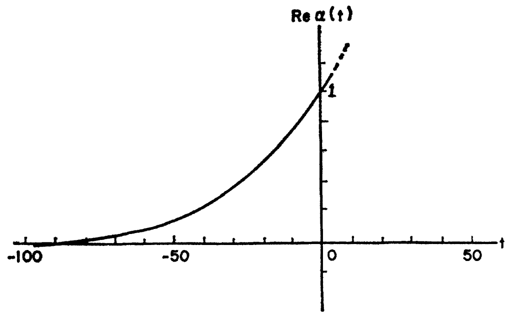

FIG. 2. Plot of the real part of the complex angular momentum as a function of t, where t is the squared-momentum transfer in the s channel ($t\leqslant 0$) or the square of the total energy in the t channel with the exchanged mass $a=4$. The dashed portion of the curve is based on a rough estimate.

It is not convenient to use Eq. (10) to obtain the Regge trajectory since the right-hand side also depends on α.

To the order of g^2, one has from Eq. (10) that

$$\alpha=-1+\frac{g^2}{32\pi^2}\int_{-1}^{1}dz\frac{1}{-\frac{1}{4}t(1-z^2)+m^2}.\quad(11)$$

Equation (11) is identical to Eq. (29) of reference 4. The trajectory given by Eq. (11) goes to infinity at threshold, $t=4m^2$, a behavior which is characteristic of this order of perturbation theory. However, the second term in the denominator of Eq. (10), which contains the higher-order terms of g^2, removes the divergence in Eq. (11) and leads to a smooth trajectory across the region $t\approx 4m^2$. Note that Eq. (11) is independent of the exchanged squared mass a.

In order to obtain the trajectory, put $f(s_0)=\delta(s_0-a)$ in Eq. (5). The result in units $m^2=1$ is

$$1=\frac{g^2}{32\pi^2}\int_0^1 dx\,\frac{x^\alpha}{1-x}I(t,x,a),\quad(12)$$

where

$$I(t,x,a)=\int_{-1}^{1}\frac{dz}{-\frac{1}{4}t(1-z^2)+1+ax/(1-x)^2}.\quad(13)$$

The integration of Eq. (12) is carried out separately for the regions $t\leqslant 0$, $0\leqslant t\leqslant 4$, and $t\geqslant 4$. In order to illustrate the behavior of the Regge trajectory, the numerical plot for the case $a=4$ is given in Fig. 2. The plot for other values of a can be treated similarly. When the exchanged squared mass a is increased, one finds that a stronger coupling strength yields the same $\alpha(0)$. The coupling constant is fixed so that $\alpha(0)=1$. This is consistent with constant total cross section at high energies. The value of $\alpha(t)$ is zero at $t\approx -90$. Rough estimates for $t\geqslant 4$ indicate that $\mathrm{Re}\,\alpha(t)$ is continuous and that $\mathrm{Im}\,\alpha(t)$ develops slowly.

4. THE CASE $t=0$

The case of $t=0$ is singled out because the total cross section at high energy is related to $\alpha(0)$ and the calculations can be carried out in closed form. For $a=4$, $t=0$, Eqs. (6), (10), and (11), respectively, become

$$\frac{16\pi^2}{g^2}=\int_0^1 dx\frac{x^{\alpha(0)}-x^{\alpha(0)+1}}{(1+x)^2}, \tag{14}$$

$$\alpha(0)+1=\frac{g^2/16\pi^2}{1-(g^2/16\pi^2)\sum_{n=1}^{\infty}\{(-1)^n(2n+1)/[n+\alpha(0)+1]\}} \tag{15}$$

$$\alpha(0)+1=g^2/16\pi^2. \tag{16}$$

Equations (14) and (15), which relate $\alpha(0)$ to g^2, are equivalent. A larger value of g^2 corresponds to a larger value of $\alpha(0)$. If one sets $\alpha(0)=1$, then Eqs. (14) and (15) both lead to $g^2/16\pi^2=12.6$, but Eq. (16) leads to $g^2/16\pi^2=2$. This difference in the value found for g^2 is due to the second term in the denominator of Eq. (15). For $\alpha(0)=1$, we find from Eq. (10) that $g^2/16\pi^2$ decreases smoothly from 12.6 to 0 as a is reduced from 4 to 0 as noted in Sec. 2. When $a\approx 1$ the Eqs. (14) to (16) lead to the same value $g^2/16\pi^2=2$. For $\alpha(0)>-1$, $g^2/16\pi^2$ decreases similarly from an upper bound restricted by the value of $\alpha(0)$ to 0 as a is reduced from a certain value to 0. For the case $a=0$ which corresponds to an exchange of scalar photons, the only possible value of g^2 is zero independent of the value of $\alpha(0)$.

5. SUMMARY

The unitarity and analyticity of the S matrix[6] imply $\alpha(0)\leqslant 1$. In this specific model, this condition puts an upper bound on the possible values of the coupling strength. Specifically, $\alpha(0)\leqslant\alpha_{\max}(0)=1$ leads to $g^2\leqslant(g^2)_{\max}$, because the integral in Eq. (14) is positive definite. Thus, the existence of the Pomeranchuk pole with $\alpha(0)=\alpha_{\max}(0)$ implies the existence of an interaction with $g^2=(g^2)_{\max}$. This fact in turn supports the idea that the strength of an interaction is as strong as possible, as suggested by Chew and Frautschi.[1,7]

In our approximation, all the ladder diagrams in which the particle momenta k^2 and k'^2 in $A(s',t,k^2,k'^2)$ are on the mass shell have been summed to all orders in g^2. In the language of perturbation theory, our g^2 approximation is rigorous but our g^4 and higher-order terms are approximate results of the ladder approximation.

To order g^2, the trajectory is given by Eq. (11). This equation fails in the region $t\approx 4$, because the integral diverges there. On the other hand, higher-order terms in g^2 included in Eq. (5) lead to a reasonable Regge trajectory in the whole range of values of t. At $t=-\infty$, α is -1. Beyond $t=4$, $\alpha(t)$ is complex and $\mathrm{Re}\,\alpha(t)$ approaches -1 as t goes to $+\infty$. Thus, the Regge trajectory obtained here resembles that from potential scattering.[8] This behavior of the present crude model suggests that it might be possible to extend the ideas of Regge into relativistic field theory.

Note added in proof. Equality of Eq. (40) of reference 4 with Eq. (5) can be shown by changing the variable of integration z to s' by $z=\{s'-4m^2-[4s_0x/(1-x)^2]/s'\}^{1/2}$ in Eq. (5) and then changing the order of integration. The equality was first proven by C. Goebel by finding the jump of Eq. (5) across the cut $4m^2\leqslant s<\infty$ (private communication).

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[7] We are indebted to Dr. Peter Freund for this emphasis.

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